AD-A040 679 ARMY MATERIALS AND MECHANICS RESEARCH CENTER WATERTO--ETC F/G 20/11
ON STRESS CONCENTRATION FACTOR AND DEFINITIONS OF A CRACK AND S--ETC(U)
FEB 77 R BEEUWKES

UNCLASSIFIED AMMRC-MS-77-3 NL

1 OF 1
AD
A040679

END
DATE
FILMED
7-77

AMMRC MS 77-3

12 NW

ON STRESS CONCENTRATION FACTOR AND DEFINITIONS OF A CRACK AND STRESS INTENSITY FACTOR

REINIER BEEUWKES, Jr.
MECHANICS OF MATERIALS DIVISION

February 1977

Approved for public release; distribution unlimited.

ARMY MATERIALS AND MECHANICS RESEARCH CENTER
Watertown, Massachusetts 02172

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: (14) AMMRC MS-77-3	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): (6) ON STRESS CONCENTRATION FACTOR AND DEFINITIONS OF A CRACK AND STRESS INTENSITY FACTOR	5. TYPE OF REPORT & PERIOD COVERED: (9) Final Rep[redacted]t.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) Reinier Beeuwkes, Jr	8. CONTRACT OR GRANT NUMBER(s): (12) 29p.
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Army Materials and Mechanics Research Center, Watertown, Massachusetts 02172, DRXMR-T	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: (16) D/A Project: 1T161102AH42; AMCMS Code: 611102.11.855; Agency Accession: DA OE4697
11. CONTROLLING OFFICE NAME AND ADDRESS: U. S. Army Materiel Development and Readiness Command, Alexandria, Virginia 22333	12. REPORT DATE: (11) Feb[redacted]77
	13. NUMBER OF PAGES: 27
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Stress intensity
Stress concentration
Effective crack tip radius
Fracture
Cracks

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE REVERSE SIDE)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 403 105

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

mt

Block No. 20

(square root of r)

ABSTRACT

The common assumption that the stress is infinite at the tip of an in-plane crack is inconsistent with the basic historical solutions for stress for cases from which crack formulae have been evolved. The latter formulae do not satisfy boundary conditions. An appropriate definition of a crack, as does one presented here, should make it obvious that such conditions are to be fulfilled and when they are, the meaning of stress intensity factor as the coefficient of a $1/\sqrt{r}$ singularity is altered. It no longer represents an infinity of stress and its connection with actual failure stress through a stress concentration factor leads to a fixed, rather than experimental, connection between Modes I and II stress intensity factors. Furthur discussion of appropriate representations of cracks for shear and normal stress loading is warranted, as well as of toughness definition consistent with failure mechanisms and with elastic-plastic solutions for stress.

CONTENTS

INTRODUCTION

This paper* is concerned with an apparent oversight in the usual current operational definitions of stress intensity factor for in-plane cracks, and in its resolution. This oversight leads to an inconsistency between the definitions of stress intensity factor for Modes I and II† as well as to a lack of definition for stress intensity factor for combinations of these modes, assuming the intent of the definitions is to make fracture stress a unique common basis. The oversight stems from a lack of satisfaction of crack boundary conditions. This, in turn, derives from setting the minor axis of the Inglis solution equal to zero, in evolving the formulation of the current definitions of stress intensity factor. The consequent aberration does not occur if, in reducing the Inglis solution, the tip radius is retained along with the crack length, even though the radius may be assigned as small (but finite) a value as one chooses. There is no infinity of stress.

Thus the radius becomes a part of our suggested definition of a crack and accordingly cracks can be regarded as notches.

Before discussing the proper deduction of stresses near crack tips, we will briefly review the current operational definitions of stress intensity factor and show that they are not inconsistent with a crack tip radius concept and, in fact, can even be regarded as a stress, not the usual stress $X\sqrt{\text{length}}$, so long as fracture stress failure of the cracked material is envisioned.

This paper is an elaboration of part of a former one [1] in which the writer gave his own definition of a crack and indicated how he had been treating cracks as notches in the prediction of failure.

On Current Definitions of K

According to current complex variable analyses which assume rectilinear elasticity, the stress intensity factors, K, for Modes I and II are determined as the coefficient of a $1/\sqrt{2r}$ singularity by the formula

$$K_I - iK_{II} = 2\sqrt{2} \lim Z^{1/2} \phi'(Z) \qquad Z \to 0 \qquad (1)$$

in which ϕ is a stress function whose expansion commences with a $\sqrt{Z}$ term. r is the radial distance from the crack tip.

We now consider Mode I, i.e., crack axis is perpendicular to the direction of the resultant loading.

The above definition really stems from taking K to be the coefficient of $(2r)^{-1/2} f(\phi)$ in the expressions for the rectangular components of stress as distance, r, from the tip of the crack approaches zero, i.e., typically‡

$$S = \frac{K}{\sqrt{2r}} f(\theta) \qquad (2)$$

*Manuscript received 3 June 1975.

†Mode I: Loading stresses at infinity perpendicular to and along the crack axis; Mode II: Loading stresses at infinity are shearing stresses along and perpendicular to the crack axis.

‡The complete expressions may be found on pages 7 and 10.

1. BEEUWKES, R., Jr. *Analysis of Failures.* Proceedings of the Third Sagamore Ordnance Materials Research Conference, December 1956, co-sponsored by the Ordnance Materials Research Office (now Army Materials and Mechanics Research Center) and the Office of Ordnance Research of the U. S. Army, p. 99.

which to be dimensionally correct, and because stress is proportional to load, may also be written*

$$S = \frac{(k/2)S_n\sqrt{a}}{\sqrt{2r}} \; f(\theta)$$

i.e.,

$$K = (k/2)\; S_n\sqrt{a}$$

where S_n is a nominal loading stress†, "a" is a length which directly or indirectly characterizes crack size and k is a function of crack specimen geometry. Thus for the standard reference case, a crack in a plate many crack lengths long and wide, and loaded cross-wise to the crack by a uniform end loading stress S_n, k = 2 if "a" is half the crack length. If this plate were reduced symmetrically to width w, keeping the crack in the center, $k \cong 2 (\cos \pi a/w)^{-1/2}$. If the original plate had been sectioned along the specimen axis, cutting the crack in half, and the meaning of S_n and "a" were retained, k = (2X1.12147). If instead of the center crack our original specimen had two very (~ infinitely) deep cracks symmetrically situated to each other on opposite sides of, and perpendicular to the specimen axis with the nominal stress conveniently selected to be the axial load divided by the net section area between the crack tips and "a" selected to be half the distance between the tips, then $k = 4/\pi$.

This formula for stress purports to uniquely depict the stress field immediately about the crack tip; k embodies the effect of specimen geometry and load distribution but not the distribution of stress at the tip.

Clearly the formula may also be written, by dividing both numerator and denominator by $\sqrt{\rho}$,

$$S = \frac{K/\sqrt{\rho}}{\sqrt{2\,r/\rho}} \; f(\theta) = \frac{K_o}{\sqrt{2\,r/\rho}} \; f(\theta) \tag{3}$$

in which we see that distance is now measured in terms of a characteristic length, such as a crack tip radius, i.e., distance is now measured by r/ρ.

Also

$$K_o = K/\sqrt{\rho} = (k/2)\; S_n\sqrt{a/\rho} \tag{4}$$

a quantity which has the dimension of a stress and an appearance like that of the concentrated stress at the base of a notch. It could be computed if ρ were a known quantity.

---

*k/2 is written as $Y/\sqrt{\pi}$ in some literature (Y depends on the dimension chosen for "a"). The reason for choosing k/2 for the constant, instead of simply k, is to identify k with the stress concentration factor coefficient of $\sqrt{a/\rho}$ as will be seen later on. Also, our $K\sqrt{\pi}$ is often called K, as a result of an energy approach, but is redundant and distracting in the formulae for stresses.

†In determining K for a material, S_n and "a" are measured at a failure load defined by some Standard, especially the ASTM Standard for toughness testing.

Now the term "stress intensity factor" when used to evaluate toughness of materials may be taken to imply that a fracture stress, $S = F$, is attained at failure according to the formula for K involving stress and crack length evaluated at failure. That is, since $f(\theta) = 1$ when S is a maximum,

$$S = F = K_o/\sqrt{2r/\rho} \tag{5}$$

where, to have similarity at failure in different tests, r/ρ = constant. If, for example, $r/\rho = 1/2$,

$$F = K/\sqrt{\rho} = K_o$$

or, if F and K are known experimentally

$$\rho = (K/F)^2$$

Instead of K, G is sometimes used, where, in our first example referring to a central crack in a tension member

$$S_n = \sqrt{\frac{GE}{\pi a(1-\mu^2)}} \tag{6}$$

and G corresponds to the failure value of S_n when the crack length is 2a. Here, in analogy to the Griffith formula,

$$S_n = \sqrt{\frac{2}{\pi} \frac{E\gamma}{a(1-\mu^2)}}$$

for crack failure when surface energy is controlling, writers assume G represents the energy absorbed at crack failure, per unit length of crack growth.

This assumption is unwarranted if G and K are computed from the same critical values of S_n and "a". While an energy criterion must be met, it is not necessarily controlling. However, assuming G is controlling, we have

$$\frac{EG}{\pi(1-\mu^2)} = K^2 \tag{7}$$

and

$$\rho = \left(\frac{K}{F}\right)^2 = \frac{EG}{\pi(1-\mu^2)F^2}$$

Definition of a Crack [1]

In accordance with our previous remarks, we assume that for the mathematical treatment of macroscopic cracks, the crack may be regarded as a symmetrical notch with a smoothly varying curvature and a parabolic tip of sufficiently small curvature compared with its depth. "Sufficiently small curvature" means that an elasticity stress analysis would show that the peak stress in the region of the tip, is inversely proportional to the square root of the tip radius of curvature to any close approximation desired. Since any curve with continuous second derivatives is characterized at a point by its curvature, it will be seen that this definition includes more than obviously crack-like ellipses and hyperbolae.

This definition clearly requires satisfaction of boundary conditions at the tip of the crack. This is not true of the usual conception of a line crack (at least in the way it has been treated). In the latter case any symmetrical self-equilibrating combination of concentrated loads could be applied to the tip along the crack axis, and satisfy the boundary conditions as well as does the analysis one has come to expect, which makes the stress in the crack axis direction equal to that perpendicular to the crack axis, instead of zero* as it should be and would be in the case of the parabolically tipped crack. These loads would also cause stress to decrease with distance from the crack tip at a different rate than the inverse square root of this distance, in contrast to the current treatments.

This definition excludes, or relegates to separate treatment, a crack with parallel sides and circular tip. It and other shapes should be studied separately not only for pedagogical reasons, but because they may conceivably be encountered, or prove useful experimentally. With them peak stresses are reached at different loads and amount of yielding near the crack tips. However, the definition adopted here derives historically, as do the conventional crack formulae for K, from the well-known Inglis classical solution for stresses about elliptical holes and thus may be readily visualized and the Inglis solution utilized by the reader in deriving the results to be presented below. The definition is however, not only suitable for present purposes, but separate investigation [2], not presented here, indicates that its use is not limited to smooth contours but that the material to which it is applied may be considered to have an effective radius when there is some plasticity about the crack tip, an effective radius which is a material property and is associated with failure below the crack tip with its irregular contour.

Stress Formulae with Parabolic Crack Tip

The general formulae for stress components, in the vicinity of the crack tip may, as previously indicated, conveniently be deduced from the Inglis solution for stresses about an elliptical crack, if this solution is expressed in terms of the semi-major axis "a" and the radius of curvature at its tip, ρ, instead of "a" and the semi-minor axis "b". That is, for "b" we first substitute $b = \sqrt{a\rho}$, then replace the coordinate system by a natural one that is near the tip and finally reduce the resulting formulae to the limiting case where $a/\rho >> 1$ and distance from the tip, measured in tip radii, is small.

The resulting formulae for the rectangular components are as follows. The corresponding formulae for the polar and parabolic coordinate components are submitted in the Appendix. The Airy stress function is

* Assuming no internal pressure in the crack

2. BEEUWKES, R., Jr. *Characteristics of Crack Failure.* Surfaces and Interfaces, Syracuse University Press, Syracuse, N.Y., v. II, 1968, p. 277.

$$F_I = (k_I S_I/2) \left[-\frac{(\rho/a)\cos(\theta/2)}{(2\,r/a)^{3/2}} + \frac{\cos^3 \theta/2}{(2\,r/a)^{1/2}} \right] \quad (8)$$

Because of the ρ term, the stresses do not decrease with distance, r, from the tip as $r^{-1/2}$.

The origin of rectangular coordinates (x,y) and polar coordinates (r, θ) is at a distance $\rho/2$ behind the nose of the crack-like notch, on the x axis. ρ is the radius of curvature of the tip of the nose. x is along the crack axis the negative direction lying completely within the crack. The positive y axis is counter-clockwise from the positive x axis. The polar coordinate angle, θ, is positive in the direction from positive x to positive y. The coordinates (x/a), (y/a) and (r/a) are oriented and centered in the same way as (x, y) and r.

The equation of the parabola is, in polar coordinates,

$$(r/\rho)\ (2\cos^2\theta/2) = (r/\rho)\ (1+\cos\theta) = 1 \quad (9)$$

and in rectangular coordinates,

$$x/\rho = (1/2)\ [1 - (y/\rho)^2]$$

from which it is, indeed, evident that the tip is at a distance of $x = \rho/2$ in front of the origin. It is also obvious that with distance measured in tip radii, all parabolae have the same size and appear crack-like if $x >> \rho$.

Mode I: "Normal Stress" – Crack Tip Stress Symmetric with Respect to Crack Axis.

Only the loading stress perpendicular to the axis appears in the expressions for stress near the crack tip if $a/\rho >> 1$, as assumed here.

$$\frac{2\,S_x}{k_I S_I} = -\frac{(\rho/a)\cos 3\,\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos\theta/2(1-\sin\ \theta/2\ \sin 3\theta/2)}{(2r/a)^{1/2}} \quad (10)$$

$$\frac{2\,S_y}{k_I S_I} = +\frac{(\rho/a)\cos 3\,\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos\ \theta/2(1+\sin\theta/2\ \sin 3\theta/2}{(2\,r/a)^{1/2}}$$

$$\frac{2\,S_{xy}}{k_I S_I} = -\frac{(\rho/a)\sin 3\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos\theta/2\ (\sin\theta/2\cos 3\theta/2)}{(2\,r/a)^{1/2}}$$

The stresses on the boundary, $\rho/a = (2\,r/a)\cos^2\theta/2$, are, keeping the contributions of each term separate before adding them up,

$$\frac{2\,S_x}{k_I\,S_I} = -\,(\cos^3\theta/2\ \cos\,3\theta/2)\sqrt{a/\rho}\ +\ \cos^2\theta/2\,(1-\sin\theta/2\,\sin 3\theta/2)\sqrt{a/\rho} \qquad (11)$$

$$= +\,(2\sin^2\theta/2\ \cos^2\theta/2)\sqrt{a/\rho}$$

$$\frac{2\,S_y}{k_I\,S_I} = +\,(\cos^3\theta/2\ \cos 3\theta/2)\sqrt{a/\rho}\ +\ \cos^2\theta/2\,(1+\sin\theta/2\,\sin 3\theta/2)\sqrt{a/\rho}$$

$$= (2\cos^4\theta/2)\sqrt{a/\rho}$$

$$\frac{2\,S_{xy}}{k_I S_I} = -\,(\cos^3\theta/2\,\sin 3\ \theta/2)\sqrt{a/\rho}\ +\ \cos^2\theta/2\,(\sin\theta/2\,\cos 3\theta/2)\sqrt{a/\rho}$$

$$= (-2\sin\theta/2\,\cos^3\theta/2)\sqrt{a/\rho}$$

However, the stresses* in the directions u, v of parabolic coordinates,

$$u^2 = (2\,r/a)\cos^2\theta/2,\ v^2 = (2\,r/a)\sin^2\theta/2,\ \text{boundary}\ \rho/a = (2\,r/a)\cos^2\theta/2 \qquad (12)$$

are more useful in determining the relative contributions to the boundary stress state.

These are

$$\frac{2\,S_u}{k_I\,S_I} = \frac{-\rho/a\,\cos\,\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos^3\theta/2}{(2\,r/a)^{1/2}} \qquad (13)$$

so that on the boundary

$$\frac{2\,S_u}{k_I S_I} = (-\cos^4\theta/2)\sqrt{a/\rho} + (\cos^4\theta/2)\sqrt{a/\rho} = 0$$

$$\frac{2\,S_v}{k_I S_I} = \frac{\rho/a\,\cos\,\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos\theta/2\,(2-\cos^2\theta/2)}{(2\,r/a)^{1/2}}$$

* The following formulae are conveniently derived using parabolic coordinates. However, they may be found from the rectangular components of stress by taking account of the fact that in an elementary right triangle with hypotenuse ds along the parabolic boundary the angle between dy and ds is $\theta/2$.

so that on the boundary

$$\frac{2\,S_v}{k_I S_I} = (\cos^4\,\theta/2)\sqrt{a/\rho} + (2\cos^2\,\theta/2 - \cos^4\,\theta/2)\sqrt{a/\rho}$$

$$= (2\cos^2\,\theta/2)\sqrt{a/\rho}$$

$$\frac{2\,S_{uv}}{k_I\,S_I} = \frac{-\rho/a\,\sin\,\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos^2\,\theta/2\,\sin\,\theta/2}{(2\,r/a)^{1/2}}$$

so that on the boundary

$$\frac{2\,S_{uv}}{k_I\,S_I} = (-\cos^3\,\theta/2\,\sin\,\theta/2)\sqrt{a/\rho} + (\cos^3\,\theta/2\,\sin\,\theta/2)\sqrt{a/\rho} = 0$$

Here again, the contributions of the two types of terms have been kept separate before adding them up.

In expressions for S_y and S_v for $\theta = 0$, we recognize the usual formula for stress concentration of a very sharp elliptical notch, $2\sqrt{a/\rho}$, where $k_I = 2$ as it is for that case when S_I is the loading stress far from the notch.

As stated previously, the first term was unfortunately omitted in the definition of stress intensity factor so that the formulae for stresses were written

$$S_x = \frac{K_I\,\cos\,\theta/2\,(1-\sin\,\theta/2\,\sin\,3\,\theta/2)}{(2\,r)^{1/2}} \tag{14}$$

$$S_y = \frac{K_I\,\cos\,\theta/2\,(1+\sin\,\theta/2\,\sin\,3\,\theta/2)}{(2\,r)^{1/2}}$$

$$S_{xy} = \frac{K_I\,\cos\,\theta/2\,(\sin\,\theta/2\,\cos\,3\,\theta/2)}{(2\,r)^{1/2}}$$

Thus if these latter formulae were evaluated on the boundary, at $\theta = 0$ where the stress is maximum, the term retained in them contributes but half the true stress which is given by the prior formulae. However, comparing these sets of formulae, it is evident that

$$\frac{k_I\,S_I}{2}\sqrt{a} = K_I \tag{15}$$

and hence the expression for K_I computed mathematically as the coefficient of $(2\ r)^{-1/2}$ can be used to compute the coefficient of stress concentration, k_I, by this relation, without error, even though an improper expression for stress has been used in defining K.

If S is assumed to be F, where F is the "nil-ductility" tensile fracture stress and $K = K_{Ic}$

$$F = k_I S_I \sqrt{a/\rho} = 2\ K_{Ic}/\sqrt{\rho} \tag{16}$$

by the above relation, i.e.,

$$\rho = (2\ K_{Ic}/F)^2$$

This ρ could be a machined in radius and the stress distribution might be rectilinearly elastic up to S = F, so that this case would be one of classic brittle failure. On the other hand, this ρ might be a unique characteristic length obtained by this formula from any crack K_{Ic} test, a length which is in some cases unequivocally a crack tip radius.

Mode II: "Shear" – Crack Tip Stress Anti-Symmetric With Respect to Crack Axis

The resultant component of the shear loading which is parallel to the crack axis, is here regarded as a clockwise couple whose forces lie symmetrically above and below the crack axis, so that the force above points away from the tip in the + x direction.

The Airy stress function is

$$F_{II} = (k_{II}\ S_{II}) \left[-\frac{(\rho/a \sin \theta/2)}{(2\ r/a)^{3/2}} + \frac{3 \sin \theta/2 \cos^2 \theta/2}{(2\ r/a)^{1/2}} \right] \tag{17}$$

The stresses are

$$\frac{S_x}{k_{II}\ S_{II}} = +\frac{(\rho/a) \sin 3\theta/2)}{(2\ r/a)^{3/2}} - \frac{\sin \theta/2\ (2 + \cos \theta/2 \cos 3\ \theta/2)}{(2\ r/a)^{1/2}} \tag{18}$$

$$\frac{S_y}{k_{II}\ S_{II}} = -\frac{(\rho/a) \sin 3\ \theta/2}{(2\ r/a)^{3/2}} + \frac{\sin \theta/2\ (\cos \theta/2 \cos 3\ \theta/2)}{(2\ r/a)^{1/2}}$$

$$\frac{S_{xy}}{k_{II}\ S_{II}} = -\frac{(\rho/a) \cos 3\ \theta/2}{(2\ r/a)^{3/2}} + \frac{\cos \theta/2\ (1 - \sin \theta/2 \sin 3\ \theta/2)}{(2\ r/a)^{1/2}}$$

As was the case with Mode I, the stresses in the directions u, v of parabolic coordinates are more useful in determining the relative contributions to the boundary stress state.

These are

$$\frac{S_u}{k_{II}\,S_{II}} = \frac{(\rho/a)\ \sin\ \theta/2}{(2\ r/a)^{3/2}} - \frac{\sin\ \theta/2\ \cos^2\ \theta/2}{(2\ r/a)^{1/2}} \tag{19}$$

so that on the boundary

$$\frac{S_u}{k_{II}\,S_{II}} = \cos^3\theta/2\ \sin\ \theta/2\ \sqrt{a/\rho} - \sin\ \theta/2\ \cos^3\ \theta/2\ \sqrt{a/\rho} \quad = 0$$

$$\frac{S_v}{k_{II}\,S_{II}} = -\frac{(\rho/a)\ \sin\ \theta/2}{(2\ r/a)^{3/2}} - \frac{\sin\ \theta/2\ (1 + \sin^2\ \theta/2)}{(2\ r/a)^{1/2}}$$

so that on the boundary

$$\frac{S_v}{k_{II}\,S_{II}} = -\cos^3\ \theta/2\ \sin\ \theta/2\ \sqrt{a/\rho} - (2\cos\theta/2\ \sin\ \theta/2 - \cos^3\theta/2\sin\theta/2)\sqrt{a/\rho}$$

$$= -\frac{\sin\theta/2\ \cos^2\theta/2}{2}\sqrt{a/\rho} - \frac{\sin\theta\ (2 - \cos^2\theta/2)}{2}\sqrt{a/\rho}$$

$$= -\ \sin\ \theta\sqrt{a/\rho}$$

$$\frac{S_{uv}}{k_{II}\,S_{II}} = -\frac{(\rho/a)\ \cos\ \theta/2}{(2\ r/a)^{3/2}} + \frac{\cos^3\ \theta/2}{(2\ r/a)^{1/2}}$$

so that on the boundary

$$\frac{S_{uv}}{k_{II}\ S_{II}} = -\ \cos^4\ \theta/2\ \sqrt{a/\rho} + \cos^4\ \theta/2\ \sqrt{a/\rho} \quad = 0$$

As before, the contributions of the two types of terms have been kept separate before adding them up.

In the expression for S_v we recognize the formula for stress concentration of a very sharp elliptical notch $\mp\sqrt{a/\rho}$, corresponding to extremes of $\sin\theta$, $\theta = \pm\,\pi/2$, where $k_{II} = 1$ as it is for the case when S_{II} is the loading stress far from the notch.

Again, as we observed previously, the first term was unfortunately omitted in the definition of stress intensity factor so that the formulae for stresses were written

$$S_x = -\frac{K_{II} \sin \theta/2\ (2 + \cos\theta/2\ \cos 3\theta/2)}{(2\,r)^{1/2}} \tag{20}$$

$$S_y = +\frac{K_{II} \sin \theta/2\ (\cos \theta/2\ \cos 3\theta/2)}{(2\,r)^{1/2}}$$

$$S_{xy} = +\frac{K_{II} \cos \theta/2\ (1 - \sin \theta/2\ \sin 3\theta/2)}{(2\,r)^{1/2}}$$

Thus if the latter formula were evaluated for stress along the boundary $\rho = 2\,r \cos^2 \theta/2$, we would have

$$S_v = S_y \cos^2 \theta/2 + S_x \sin^2 \theta/2 - 2\, S_{xy} \sin \theta/2 \cos \theta/2 \tag{21}$$

i.e.,

$$\frac{S_v\sqrt{\rho}}{k_{II}} = -\frac{\sin\theta}{2}\Big\{-\cos^3 \theta/2 \cos 3\theta/2) + 2 \sin^2 \theta/2$$

$$+ \sin^2 \theta/2 \cos \theta/2 \cos 3\theta/2)$$

$$+ 2\cos^2 \theta/2 - 2\sin \theta/2 \cos^2 \theta/2 \sin 3\theta/2)\Big\}$$

$$= -\frac{\sin\theta}{2}\Big\{2 - 2\cos^3 \theta/2 \cos 3\theta/2 + \cos \theta/2 \cos 3\theta/2)$$

$$- 2 \sin \theta/2 \sin 3\theta/2 \cos^2 \theta/2\Big\}$$

$$= -\frac{\sin\theta}{2}\Big\{2 + \cos \theta/2 \cos 3\,\theta/2 - 2\cos^2 \theta/2 \cos\theta\Big\}$$

$$= -\frac{\sin\theta}{2}\Big\{2 + \cos \theta/2\ \big[4\cos^3 \theta/2 - 3\cos\theta/2 - 4\cos^3 \theta/2 + 2\cos\theta/2\big]\Big\}$$

$$= -\frac{\sin\theta}{2}\Big\{2 - \cos^2 \theta/2\Big\} = -\frac{\sin\theta}{4}\Big\{3 - \cos\theta\Big\}$$

Here we have the same expression that appeared above as the coefficient of $\sqrt{a/\rho}$ in the second term of S_v in parabolic coordinates. This expression has a maximum at $\theta = \pi/2 + \cos^{-1} \{(3 - \sqrt{17})/4\}$ but this is misleading. The true extremes are at $\mp \theta = \pi/2$ as was seen in the total expression for S_v Thus the above expression is $\mp 3/4$ at $\theta = \pm \pi/2$, instead of ∓ 1, the value of the complete expression.

Unfortunately for consistency of K in representing stress, this differs from the ratio of 1 to 2 found for Mode I.

For Mode II, it appears, in comparing the above set of formulae expressed in terms of K_{II} with that satisfying the boundary conditions, that

$$k_{II} S_{II} \sqrt{a} = K_{II} \tag{22}$$

and hence the expression for K_{II} computed mathematically as the coefficient of $(2\,r)^{-1/2}$ can be used to compute the coefficient of stress concentration, k_{II}, by this relation, without error, even though an improper expression for stress has been used in defining K, an expression which does not even specify the angle θ at which maximum stress occurs.

Again, if S is assumed to be F, where F is the nil ductility tensile fracture stress and $K_{II} = K_{IIc}$,

$$F = k_{II} S_{II} \sqrt{a/\rho} = k_{IIc}/\sqrt{\rho} \tag{23}$$

by the above relationship of $k_{II} S_{II} \sqrt{a}$ to K_{II}.

Therefore

$$\rho = (K_{IIc}/F)^2$$

As stated with reference to Mode I, thus ρ could be a machined-in radius and the stress distribution might be rectilinearly elastic up to S = F so that this case would be one of classic brittle failure. On the other hand this ρ might be a unique characteristic length computed by this formula from any crack K_{IIc} test, a length which is in some cases unequivocally a crack tip radius.

Combination of Modes I and II: Maximum Boundary Stress

We take axes and loading stresses, S_I and S_{II} as discussed under Modes I and II. Assume $S_I \geqslant 0$, $S_{II} \geqslant 0$.

Then, from the results already found for the boundary stress, $S_u = S_{uv} = 0$ and

$$S_v = (1+\cos\theta)\frac{k_I S_I}{2}\sqrt{a/\rho} - (\sin\theta)\; k_{II} S_{II}\sqrt{a/\rho} \tag{24}$$

which is clearly an extreme when

$$\sin\theta/\cos\theta = -k_{II}\,S_{II}/(k_I\,S_I/2);\ \text{thus } -\pi/2 \leqslant \theta \leqslant 0$$

Thus, substituting for sin θ,

$$S_{v,\,ext} = \left\{ (1+\cos\theta)\,\frac{k_I\,S_I}{2} + \frac{(k_{II}\,S_{II})^2}{k_I\,S_I/2}\cos\theta \right\}\sqrt{a/\rho} \tag{25}$$

and noting from the extremum condition that

$$\sin^2\theta\ (= 1 - \cos^2\theta) = [k_{II}\,S_{II}/\,(k_I\,S_I/2)]^2\cos^2\theta$$

so that

$$\cos\theta = \frac{k_I\,S_I/2}{\sqrt{(k_I\,S_I/2)^2 + (k_{II}\,S_{II})^2}}$$

we have

$$S_{v,ext} = \left\{ (k_I\,S_I/2) + \sqrt{(k_I\,S_I/2)^2 + (k_{II}\,S_{II})^2} \right\}\sqrt{a/\rho}$$

This clearly reduces to $k_I\,S_I\sqrt{a/\rho}$, 0 if S_{II} is zero and $+\,k_{II}\,S_{II}\sqrt{a/\rho}$ if S_I is zero, as it should.

Since

$$K_I = \frac{k_I\,S_I\sqrt{a}}{2};\quad K_{II} = k_{II}\,S_{II}\sqrt{a} \tag{26}$$

it may also be written

$$S_{v,ext}\sqrt{\rho} = K_I + \sqrt{K_I^2 + K_{II}^2}$$

We would certainly expect a sharply notched, $a/\rho >> 1$, isotropically brittle elastic material to fail at the notch surface when either of these expressions reach a fracture stress $S_v = F$.

Since $\rho = 2\,r\cos^2\theta/2 = r\,(1+\cos\theta)$, the radial distance to the point on the parabola where this extreme occurs is

$$\frac{r}{\rho} = \frac{1}{1+\cos\theta}$$

$$= \left[1 + (k_I S_I/2)/\sqrt{(k_I S_I/2)^2 + (k_{II}\,S_{II})^2}\right]^{-1} \tag{27}$$

measured in tip radii. From the above expressions for K_I, and K_{II}, this may also be written

$$\frac{r}{\rho} = \left[1 + K_I/\sqrt{K_I^2 + K_{II}^2}\right]^{-1} \tag{28}$$

or, if $k_I = 2\ k_{II}$, the normal case,

$$\frac{r}{\rho} = \left[1 + S_I/\sqrt{S_I^2 + S_{II}^2}\right]^{-1} \tag{29}$$

Unfortunately, in general, we cannot define or specify a mixed mode stress intensity factor from the preceding. This is because the formulas for K in terms of S_v and ρ are inconsistent for Modes I and II. That is, for given S_v and ρ, K_I and K_{II} should be the same since both refer to a stress, S_v, acting at, and tangentially to the boundary of the notch, a stress which would cause failure if it reached a unique fracture value, F, whether or not it was caused by Mode I or Mode II loading even though these cause the maximum stress (i.e., S_v = F) to be reached in different locations.

Thus

$$S_v = k_I\ S_I\ \sqrt{a/\rho} = 2\ K_I/\sqrt{\rho}$$

$$S_v = k_{II}\ S_{II}\ \sqrt{a/\rho} = K_{II}/\sqrt{\rho}$$

so that for the same S_v and ρ,

$$K_I = S_v\sqrt{\rho}\,/\,2$$

$$K_{II} = S_v\ \sqrt{\rho}$$

which K are not equal.

The notch stress factors kS, i.e., $k_I\ S_I$ and $k_{II}\ S_{II}$, do not suffer from this defect. For a fixed value of S_v and a fixed value of $\sqrt{a/\rho}$

$$k_I\ S_I = k_{II}\ S_{II} \tag{30}$$

Thus

$$kS = \left\{(k_I\ S_I/2) + \sqrt{(k_I\ S_I/2)^2 + (k_{II}\ S_{II})^2}\right\} \tag{31}$$

is the stress factor representing the combined loading S_I, S_{II}.

One may well wonder why K instead of k was introduced, or at least why the proportionality factor of K to S was not made the same for Modes I and II.

Normally, with stresses S_I and S_{II} specified as discussed previously,

$$k_I = 2\, k_{II} \tag{32}$$

so that <u>in this case</u>

$$S_{v,max} = \frac{k_I}{2} \left\{ S_I + \sqrt{S_I^2 + S_{II}^2} \right\} \sqrt{a/\rho} \tag{33}$$

$$= k_{II} \left\{ S_I + \sqrt{S_I^2 + S_{II}^2} \right\} \sqrt{a/\rho}$$

and we can define a mixed-mode stress intensity factor, $K_{I,\,II}$ in terms of an effective nominal stress, $S_{I,\,II}$, i.e.,

$$S_{I,\,II} = S_I + \sqrt{S_I^2 + S_{II}^2} \tag{34}$$

$$S_{v,\,max} = \frac{k_I}{2}\, S_{I,II} \sqrt{a/\rho} = k_{II}\, S_{I,II} \sqrt{a/\rho}$$

and

$$K_{I,II} = \frac{k_I}{2}\, S_{I,II} \sqrt{a} = k_{II}\, S_{I,II} \sqrt{a}$$

$$S_{v,\,max} = \frac{K_{I,\,II}}{\sqrt{\rho}}$$

<u>Mixed Mode Example: Tension Member with Central Elliptical Crack at an Angle to an Axis</u>

Here, if T is the nominal tensile stress,

$$S_I = T \cos^2 \theta \tag{35}$$

$$S_{II} = T \sin \theta \cos \theta$$

where θ is measured counterclockwise from the lateral axis perpendicular to the tensile axis.

Here also $k_I = 2;\ k_{II} = 1$

so that

$$K_{I,II} = \sqrt{a}\ \left\{ S_{I,II} \right\} \tag{36}$$

$$= \sqrt{a}\ \left\{ S_I + \sqrt{S_I^2 + S_{II}^2} \right\}$$

$$= T\sqrt{a}\ \left\{ \cos^2\ \theta + \sqrt{\cos^4\theta + \sin^2\ \theta\ \cos^2\ \theta} \right\}$$

$$= T\sqrt{a}\ \left\{ \cos^2\ \theta + \cos\theta \right\}$$

Thus

$$K_{I,II,max} = 2\ T\ \sqrt{a} \tag{37}$$

$$= K_I$$

Similarly,

$$S_v = \left[S_I \pm \sqrt{S_I^2 + S_{II}^2} \right] \sqrt{a/\rho} \tag{38}$$

$$= T\ \sqrt{a/\rho}\ \ [\cos\theta\ (\cos\theta \pm 1)]$$

This is maximum at $\theta = 0$, (+ sign) where

$$S_v = 2\ T\ \sqrt{a/\rho} \tag{39}$$

and at $\theta = \pi/4$ (– sign) where

$$S_v = \left(\frac{1 - \sqrt{2}}{2} \right)\ T\ \sqrt{a/\rho} \tag{40}$$

DISCUSSION

It may be observed, as a partial justification of current definitions of K, that they are proportional to stress concentration factors in the usual case that $k_{II} = k_I/2$, sic,

$$(k_I/2)\ S_I \sqrt{a} = K_I \tag{41}$$

$$k_{II}\ S_{II}\sqrt{a} = (k_I/2)\ S_{II} \sqrt{a} = K_{II}$$

so that the stress concentration factors for these two cases would be the same if ρ were the same, i.e., we would have

$$(k_I/2) \sqrt{a/\rho} = k_{II} \sqrt{a/\rho} \tag{42}$$

However, the standpoint of this paper is that this is an improper approach, that the proper one requires that $K_I = K_{II}$ if the fracture stress is fixed.

Or it might be objected that though the ρ term has the same dimensions as the $(2\ r)^{-1/2}$ term, i.e., $\rho/(2\ r)^{3/2}$ vs. $(2\ r)^{-1/2}$, that it nonetheless does diminish at a much greater rate than does the $(2\ r)^{-1/2}$ term, and therefore might be regarded as not very significant, especially in the non-homogeneous crystalline state of a crack front of metallic material. From this standpoint one might also question the $(2\ r)^{-1/2}$ term also, and thus this whole elasticity approach to the problem. Where does one decide what to retain? Such an argument, if rigorously adhered to, would ultimately question the usefulness of the macroscopic theory of elasticity, as applied to metallic materials, even in regions away from serious stress concentration. Thus in this connection we point out that even a crack front ordinarily passes many grains and hence a theory assuming homogeneity may be expected to have some practical use. The irregular front may even contribute to the concept rather than negating it; for example, there may be an effective radius due, among other possibilities, to the differences in crack length and therefore openings from point to point along the front. In fact, as indicated previously, investigation [2] [3] indicates that crack failure is generally sub-surface and that an effective radius, such as a radius consistent with the crack opening contour near the tip, is suitable and sufficient for a radius term in the calculation of the fracture stress causing failure.

However, the objection of the preceding paragraph is not germane to the main purpose of this paper, though it is to whether the elasticity treatments of K are useful or not. More on this below. The point here is the validity and self-consistency of the elasticity treatment itself, whether useful or not. That is, elasticity theory, only, is used in the formula for K which is used in K_c determinations. That theory should be cast into a form which satisfies boundary conditions. And it should embrace artificial, machined in cracks of indefinitely small, but assignable, tip radius. These factors are accounted for in the treatment in this paper, but it is well to remember also that Neuber [4] in his treatment of sharp notches, using a texture particle zone below the notch root characteristic of the material employed, took care to satisfy boundary conditions. His result was that a material constant, one half the zone depth, i.e., $\epsilon/2 = \rho'$, took the place of the tip radius ρ in the usual theory.

For self consistency the viewpoints expressed above and in the text for crack-like notches must hold for V-shaped notches, or notches whose tips are V-shaped. The crack-like notches should

3. TAGGART, R., and WAHI, K., University of Washington, Seattle. *Crack Opening Displacement During the Tensile Loading of Ductile and Brittle Notched Plates.* Private Communication, April 1971.
4. NEUBER, H. *Theory of Notch Stresses: Principles for Exact Stress Calculation.* The David W. Taylor Model Basin, Translation 74, November 1945, p. 160, and reference to "Eine neue elastiche Materialkonstante" by L. Foppl, Ing.-Archiv., v. 7, no. 4, 1936, p. 229-236.

be considered as a particular case of the V-shaped notches. In fact, the sides of the V should be considered to be the asymptotes of the actual notch shape, not the notch itself. Just as in the crack case, in order to satisfy boundary conditions one would incorporate a tip radius into the notch and would then find that no matter how small the radius is taken to be, that the boundary stress term in which it is incorporated, contains a fixed proportion of the total boundary stress. One should not therefore, take the pure V limit as the V notch anymore than the particular case of V-notch, the straight line, as the crack. From this point of view, certain well known solutions [5] [6] for open notches are incomplete, and require an additional term incorporating a radius, to complete them.

That these comments about tip radii are not unimportant will be appreciated when it is considered that one could not solve for the stresses about a hole having V-shaped corners through the use of a stress function (with undetermined coefficients) which, while assuming the corner is reached as the zero limit of a radius, assumes that the corner stress will die out in a power given by the well-known corner solutions referred to above.

A form of notch with V-shaped asymptotes, which contains the parabolic notch treated in this paper, is the following

$$r \cos^{\beta/(\pi/2)} \left[\frac{\theta}{\beta/(\pi/2)}\right] = \left[1 - (\pi/2)/\beta\right] \rho \tag{43}$$

in polar coordinates, where ρ is the radius of the tip and $\theta = \pm\beta$ defines the asymptotic opening of the notch. Thus $2(\pi-\beta)$ is the asymptotic angle of opening. The tip is at a distance

$$r = 1 - \left[(\pi/2)/\beta\right] \rho \tag{44}$$

beyond the origin of coordinates on the symmetry axis $\theta = 0$. If $\beta = \pi$, $2 r \cos^2 \theta/2 = \rho$, the crack case.

Two other points should be mentioned lest unfortunate presumptions should be drawn from the text of this paper by the reader.

One, the writer knows of no a priori reason to assume that ρ for Modes I and II are equal, as assumed here in the treatment for combined loading stresses, unless ρ is created to be so artificially, as by machining.

And finally, the writer doubts that K calculated through use of the formulae advocated here, will be found to describe the toughness of materials unless the materials are perfectly brittle (– or some presently undefined meaning or expression is assigned to ρ). That is to say, for example, that if a fracture stress were calculated by the Mode I formula from the failure load in a Mode I experiment, the load corresponding to this same fracture stress, calculated by the Mode II formula would not be the actual failure load in a Mode II experiment, unless the material was perfectly brittle.

---

5. WILLIAMS, M. L. *Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates in Extension.* Journal of Applied Mechanics, v. 19, no. 4, December 1952, p. 526.
6. WEBER, C. *Allseitig geozogene Ebene mit Zweibogenloch.* Zeitschrift für Angewandte Mathematik, und Mechanik, Band 31, Heft 7, July 1951, p. 193.

Only actual experimental results are likely to be convincing on this point, but this writer's approximate theory [2] and calculations indicate that the failures would be sub-surface and that hydrostatic stress associated with plastic deformation, sufficient to bring the stress to the failure level, is not readily attained in shear. Nonetheless, the complete theory of the elastic-plastic interaction includes the formulae of this paper as a limit.

APPENDIX

The following equations are expressions for stresses at and near ($1/2 \leq r/\rho << a/\rho$) the tips of crack like ($a/\rho >> 1$) notches in the directions of, and given in parabolic and polar coordinates. ρ is the radius of the tip of the nose and "a" is a length usually characteristic of its depth, being the semi-major axis in the standard case of an elliptically shaped crack, while r is a polar coordinate such that the tip is at $r = \rho/2$, as specified below.

The origin of the polar coordinates (r, θ) and focus of the parabolic coordinates as well as the origin of the associated rectangular coordinates, is at a distance $\rho/2$ behind the tip of the nose of the crack-like notch, on the crack axis. x is along the crack axis, the negative direction lying completely within the crack. The positive y axis is counterclockwise from the positive x axis. The polar coordinate angle θ, is positive in the direction from positive x to positive y. The coordinates $[(x/a), (y/a)]$, (r/a) are oriented and centered in the same manner as (x,y), (r,θ) and (u,v).

Polar Coordinates

Mode I

$$\frac{2\,S_r}{k_I\,S_I} = -\frac{(\rho/a)\cos\theta/2}{(2\,r/a)^{3/2}} + \frac{-\cos^3\theta/2 + 2\cos\theta/2}{(2\,r/a)^{1/2}} \qquad \text{(A1)}$$

$$\frac{2\,S_\theta}{k_I S_I} = +\frac{(\rho/a)\cos\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos^3\theta/2}{(2\,r/a)^{1/2}}$$

$$\frac{2\,S_{r\theta}}{k_I\,S_I} = +\frac{(\rho/a)\sin\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos^2\theta/2\ \sin\theta/2}{(2\,r/a)^{1/2}}$$

Mode II

$$\frac{S_r}{k_{II}S_{II}} = +\frac{(\rho/a)\ \sin\theta/2}{(2r/a)^{3/2}} - \frac{\sin\theta/2\ (3\cos\theta-1)}{2\,(2\,r/a)^{1/2}} \qquad \text{(A2)}$$

$$\frac{S_\theta}{k_{II}\,S_{II}} = -\frac{(\rho/a)\sin\theta/2}{(2\,r/a)^{3/2}} + \frac{\cos\theta/2\ (3\sin\theta)}{2\,(2\,r/a)^{1/2}}$$

$$\frac{S_{r\theta}}{k_I\ S_{II}} = +\frac{(\rho/a)\cos\theta/2}{(2\,r/a)^{3/2}} - \frac{\cos\theta/2\,(3\cos\theta-1)}{2\,(2\,r/a)^{1/2}}$$

Thus we have the following for S_v, S_θ and $S_{r\theta}$ on the boundary $(\rho/a) = (2\,r/a)\cos^2\theta/2$.

Mode I

$$\frac{2\,S_r}{k_I\,S_I} = -\cos^4 \theta/2\,\sqrt{a/\rho} + (-\cos^4\,\theta/2 + 2\,\cos^2\,\theta/2)\,\sqrt{a/\rho} \tag{A3}$$

$$= 2\,\cos^2\,\theta/2\,\sin^2\,\theta/2\,\sqrt{a/\rho}$$

$$\frac{2\,S_\theta}{k_I\,S_I} = +\,\cos^4\,\theta/2\,\sqrt{a/\rho} + (\cos^4\,\theta/2)\,\sqrt{a/\rho}$$

$$= 2\cos^4\,\theta/2\,\sqrt{a/\rho}$$

$$\frac{2\,S_{r\theta}}{k_I\,S_I} = \cos^3\,\theta/2\,\sin\,\theta/2\,\sqrt{a/\rho} + \cos^3\,\theta/2\,\sin\,\theta/2\,\sqrt{a/\rho}$$

$$= 2\cos^3\,\theta/2\,\sin\,\theta/2\,\sqrt{a/\rho}$$

Mode II

$$\frac{S_r}{k_{II}\,S_{II}} = \cos^3\,\theta/2\,\sin\,\theta/2\sqrt{a/\rho} - [\cos\,\theta/2\,\sin\,\theta/2\,(3\,\cos\,\theta-1)/2]\,\sqrt{a/\rho} \tag{A4}$$

$$= 2\,\sin^3\,\theta/2\,\cos\,\theta/2\,\sqrt{a/\rho}$$

$$\frac{S_\theta}{k_{II}\,S_{II}} = -\cos^3\,\theta/2\,\sin\theta/2\,\sqrt{a/\rho} + [\cos^2\,\theta/2\,(3\,\sin\,\theta)/2]\,\sqrt{a/\rho}$$

$$= 2\,\cos^3\,\theta/2\,\sin\,\theta/2\,\sqrt{a/\rho}$$

$$\frac{S_{r\theta}}{k_{II}\,S_{II}} = \cos^4\,\theta/2\,\sqrt{a/\rho} - [\cos^2\,\theta/2\,(3\,\cos\,\theta-1)/2]\,\sqrt{a/\rho}$$

$$= 2\,\sin^2\,\theta/2\,\cos^2\,\theta/2\,\sqrt{a/\rho}$$

Parabolic Coordinates

The crack boundary is a parabola, designated by a parameter $u = u_o$, opening to the left. It is surrounded by other u parabolae. If $u_o = 0$, the sides of the parabola are indistinguishable, though separate, i.e., we have the straight line crack (along the line $(y/a) = 0$). Position points are defined by (u, v) where the values of v designate a set of parabola, orthogonal to the u parabolae, opening to the right. Both u and v sets have one and the same common focus (origin) which is at $u = 0$, $v = 0$, and around which all the parabola pass. u^2 is equal to the radius of curvature of the tip of the nose of any u parabola, as is v^2 of any v parabola. If (ρ/a) is such a radius, the tip is $u^2/2 = (\rho/a)/2$ to the right of the focus if the radius is that of a u parabola, and is $v^2/2 = (\rho/a)/2$ to the left of the focus if the radius is that of a v parabola.

The connection between (u, v) and $[(x/a), (y/a)]$ is defined by the mapping functions (which can be incorporated into a single complex variable form).

$$(y/a) = u\, v; \quad (x/a) = (u^2 - v^2)/2 \tag{A5}$$

Thus

$$u^2 = x/a + r/a\ ; \quad v^2 = -\,x/a + r/a \tag{A6}$$

where

$$r/a = |[(x/a)^2 + (y/a)^2]^{1/2}|,$$

the radial distance to the parabola, from the focus. From these formulae, geometrical (as opposed to algebraical) construction of the parabolae is possible.

Also,

$$\left.\begin{aligned} \frac{(x/a)}{u^2} &= \frac{1}{2}\left[1 - \frac{(y/a)^2}{u^4}\right] \\ \text{i.e.,}\quad (x/\rho) &= \frac{1}{2}\left[1 - (y/\rho)^2\right] \end{aligned}\right\}\ \text{u parabolae} \tag{A7}$$

and

$$\left.\begin{aligned} \frac{(x/a)}{v^2} &= \frac{1}{2}\left[\frac{(y/a)^2}{v^4} - 1\right] \\ \text{i.e.,}\quad (x/\rho) &= \frac{1}{2}\left[(y/\rho)^2 - 1\right] \end{aligned}\right\}\ \text{v parabolae}$$

Evidently, from these expressions, "a" as a unit of length is not inconsistent with taking ρ as the unit of length at the nose.

In what follows we denote by ρ/a the dimensionless radius of the tip of the boundary curve only, i.e., from here on $\rho/a = u_o^2$.

The Mode I Airy stress function, F_I, is given by

$$\frac{2 F_I}{k_I S_I} = - u u_o^2 + u^3/3 \tag{A8}$$

and the Mode II Airy stress function, F_{II}, is given by

$$\frac{F_{II}}{k_{II} S_{II}} = \mp (v u_o^2 + v u^2) \tag{A9}$$

The stresses S_u, perpendicular to the u lines, S_v, in the direction of the u lines and the shearing stress S_{uv} are as follows.

<u>Mode I</u>

$$\frac{2 S_u}{k_I S_I} = - \frac{(\rho/a) u}{u^2 + v^2)^2} + \frac{u^3}{(u^2 + v^2)^2} \tag{A10}$$

$$\frac{2 S_v}{k_I S_I} = \frac{(\rho/a) u}{(u^2 + v^2)^2} + \frac{u (u^2 + 2 v^2)}{(u^2 + v^2)^2}$$

$$\frac{2 S_{uv}}{k_I S_I} = - \frac{(\rho/a) v}{(u^2 + v^2)^2} + \frac{v u^2}{(u^2 + v^2)^2}$$

Mode II

$$\frac{S_u}{k_{II}\,S_{II}} = \pm \frac{(\rho/a)\,v}{(u^2+v^2)^2} \mp \frac{v\,u^2}{(u^2+v^2)^2} \qquad \text{(A11)}$$

$$\frac{S_v}{k_{II}\,S_{II}} = \mp \frac{(\rho/a)\,v}{(u^2+v^2)^2} \mp \frac{v\,(u^2+2v^2)}{(u^2+v^2)^2}$$

$$\frac{S_{uv}}{k_{II}\,S_{II}} = \mp \frac{(\rho/a)\,u}{(u^2+v^2)^2} \pm \frac{u^3}{(u^2+v^2)^2}$$

The top signs here refer to positive θ angles and the lower ones to minus θ angles.

Thus on the boundary of the parabola $u = u_o = |(\rho/a)^{1/2}|$, the stresses are as follows.

Mode I

$$\frac{2\,S_u}{k_I\,S_I} = \frac{-(\rho/a)^{3/2}}{(\rho/a+v^2)^2} + \frac{(\rho/a)^3}{(\rho/a+v^2)^2} \qquad \text{(A12)}$$

$$= 0$$

$$\frac{2\,S_v}{k_I\,S_I} = +\frac{(\rho/a)^{3/2}}{(\rho/a+v^2)^2} + \frac{(\rho/a)^{1/2}\,(\rho/a+2\,v^2)}{(\rho/a+v^2)^2}$$

$$= \frac{2\,(\rho/a)^{1/2}}{(\rho/a+v^2)}$$

$$= 2\sqrt{a/\rho} \quad \text{maximum, when } v = 0$$

$$\frac{2\,S_{uv}}{k_I\,S_I} = -\frac{(\rho/a)\,v}{(\rho/a+v^2)^2} + \frac{(\rho/a)\,v}{(\rho/a+v^2)^2}$$

$$= 0$$

Army Materials and Mechanics Research Center,
Watertown, Massachusetts 02172
ON STRESS CONCENTRATION FACTOR AND
DEFINITIONS OF A CRACK AND STRESS INTENSITY -
Reinier Beeuwkes, Jr.

Monograph Series AMMRC MS 77-3, February 1977, 27 pp,
D/A Project 1T161102AH42, AMCMS Code 611102.11.855

AD____________

UNCLASSIFIED
UNLIMITED DISTRIBUTION

Key Words

Stress intensity
Stress concentration
Effective crack tip radius

The common assumption that the stress is infinite at the tip of an in-plane crack is inconsistent with the basic historical solutions for stress for cases from which crack formulae have been evolved. The latter formulae do not satisfy boundary conditions. An appropriate definition of a crack, as does one presented here, should make it obvious that such conditions are to be fulfilled and when they are, the meaning of stress intensity factor as the coefficient of a $1/\sqrt{r}$ singularity is altered. It no longer represents an infinity of stress and its connection with actual failure stress through a stress concentration factor leads to a fixed rather than experimental, connection between Modes I and II stress intensity factors. Further discussion of appropriate representations of cracks for shear and normal stress loading is warranted, as well as of toughness definition consistent with failure mechanisms and with elastic-plastic solutions for stress.

Army Materials and Mechanics Research Center,
Watertown, Massachusetts 02172
ON STRESS CONCENTRATION FACTOR AND
DEFINITIONS OF A CRACK AND STRESS INTENSITY -
Reinier Beeuwkes, Jr.

Monograph Series AMMRC MS 77-3, February 1977, 27 pp,
D/A Project 1T161102AH42, AMCMS Code 611102.11.855

AD____________

UNCLASSIFIED
UNLIMITED DISTRIBUTION

Key Words

Stress intensity
Stress concentration
Effective crack tip radius

The common assumption that the stress is infinite at the tip of an in-plane crack is inconsistent with the basic historical solutions for stress for cases from which crack formulae have been evolved. The latter formulae do not satisfy boundary conditions. An appropriate definition of a crack, as does one presented here, should make it obvious that such conditions are to be fulfilled and when they are, the meaning of stress intensity factor as the coefficient of a $1/\sqrt{r}$ singularity is altered. It no longer represents an infinity of stress and its connection with actual failure stress through a stress concentration factor leads to a fixed rather than experimental, connection between Modes I and II stress intensity factors. Further discussion of appropriate representations of cracks for shear and normal stress loading is warranted, as well as of toughness definition consistent with failure mechanisms and with elastic-plastic solutions for stress.

Army Materials and Mechanics Research Center,
Watertown, Massachusetts 02172
ON STRESS CONCENTRATION FACTOR AND
DEFINITIONS OF A CRACK AND STRESS INTENSITY -
Reinier Beeuwkes, Jr.

Monograph Series AMMRC MS 77-3, February 1977, 27 pp,
D/A Project 1T161102AH42, AMCMS Code 611102.11.855

AD____________

UNCLASSIFIED
UNLIMITED DISTRIBUTION

Key Words

Stress intensity
Stress concentration
Effective crack tip radius

The common assumption that the stress is infinite at the tip of an in-plane crack is inconsistent with the basic historical solutions for stress for cases from which crack formulae have been evolved. The latter formulae do not satisfy boundary conditions. An appropriate definition of a crack, as does one presented here, should make it obvious that such conditions are to be fulfilled and when they are, the meaning of stress intensity factor as the coefficient of a $1/\sqrt{r}$ singularity is altered. It no longer represents an infinity of stress and its connection with actual failure stress through a stress concentration factor leads to a fixed rather than experimental, connection between Modes I and II stress intensity factors. Further discussion of appropriate representations of cracks for shear and normal stress loading is warranted, as well as of toughness definition consistent with failure mechanisms and with elastic-plastic solutions for stress.

Army Materials and Mechanics Research Center,
Watertown, Massachusetts 02172
ON STRESS CONCENTRATION FACTOR AND
DEFINITIONS OF A CRACK AND STRESS INTENSITY -
Reinier Beeuwkes, Jr.

Monograph Series AMMRC MS 77-3, February 1977, 27 pp,
D/A Project 1T161102AH42, AMCMS Code 611102.11.855

AD____________

UNCLASSIFIED
UNLIMITED DISTRIBUTION

Key Words

Stress intensity
Stress concentration
Effective crack tip radius

The common assumption that the stress is infinite at the tip of an in-plane crack is inconsistent with the basic historical solutions for stress for cases from which crack formulae have been evolved. The latter formulae do not satisfy boundary conditions. An appropriate definition of a crack, as does one presented here, should make it obvious that such conditions are to be fulfilled and when they are, the meaning of stress intensity factor as the coefficient of a $1/\sqrt{r}$ singularity is altered. It no longer represents an infinity of stress and its connection with actual failure stress through a stress concentration factor leads to a fixed rather than experimental, connection between Modes I and II stress intensity factors. Further discussion of appropriate representations of cracks for shear and normal stress loading is warranted, as well as of toughness definition consistent with failure mechanisms and with elastic-plastic solutions for stress.

Mode II

$$\frac{S_u}{k_{II}\,S_{II}} = \pm\ \frac{(\rho/a)\,v}{(\rho/a + v^2)^2} \mp \frac{(\rho/a)\,v}{(\rho/a + v^2)^2} \qquad \text{(A13)}$$

$$= \quad 0$$

$$\frac{S_v}{k_{II}\,S_{II}} = \mp\ \frac{(\rho/a)\,v}{(\rho/a + v^2)^2} \mp \frac{v\,(\rho/a + 2\,v^2)}{(\rho/a + v^2)^2}$$

$$= \mp\ \frac{2v}{(\rho/a + v^2)}$$

$$= \mp\ \sqrt{a/\rho}, \text{ maximum, when } v^2 = \rho/a$$

$$\frac{S_{uv}}{k_{II}\,S_{II}} = \mp\ \frac{(\rho/a)^{3/2}}{(\rho/a + v^2)^2} \pm \frac{(\rho/a)^{3/2}}{(\rho/a + v^2)^2}$$

$$= \quad 0$$